

# ON THE SPEED OF RANDOM WALKS AMONG RANDOM CONDUCTANCES

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May 21st, 2012

**ABSTRACT.** We consider random walk among random conductances where the conductance environment is shift invariant and ergodic. We study which moment conditions of the conductances guarantee speed zero of the random walk. We show that if there exists  $\alpha > 1$  such that  $E[\log^\alpha \omega_e] < \infty$ , then the random walk has speed zero. On the other hand, for each  $\alpha > 1$  we provide examples of random walks with non-zero speed and random walks for which the limiting speed does not exist that have  $E[\log^\alpha \omega_e] < \infty$ .

## 1. INTRODUCTION

**1.1 Definitions and notations** We discuss the following two-dimensional model of motion in random medium. Let  $\mathbb{E}^2$  be the set of nearest neighbor edges in the graph  $\mathbb{Z}^2$ . We may also write an edge as an unordered pair  $\{x, y\}$  of neighbors in  $\mathbb{Z}^2$ . We consider the measure space  $\Omega = (0, \infty)^{\mathbb{E}^2}$ . For  $\omega \in \Omega$  and an edge  $e$ , we call  $\omega_e$  the *weight* or *conductance* of the edge  $e$ . We let  $P$  be a measure on  $\Omega$  which satisfies the following two conditions:

- (i)  $P$  is invariant and ergodic w.r.t. the group of spatial moves in  $\mathbb{Z}^2$ .
- (ii) The marginal distribution of  $\omega_e$  is the same for all choices of the edge  $e$ , i.e. vertical and horizontal edges have the same distribution.

item:veho

Note that this is weaker than invariance w.r.t. rotations. Condition (ii) can be weakened significantly, but for simplicity we keep it as is.

For a given  $\omega \in \Omega$  and  $z \in \mathbb{Z}^2$  we define the law  $P_z^\omega$  as follows.  $P_z^\omega$  is the law of a Markov chain on  $(\mathbb{Z}^2)^\mathbb{N}$  with  $P_z^\omega(X_0 = z) = 1$  and

$$P_z^\omega(X_{n+1} = y | X_n = x) = \frac{\omega(x, y)}{\sum_{w \sim x} \omega(x, w)}$$

for any neighbouring points  $x, y \in \mathbb{Z}^2$ , and where the sum in the denominator is carried over every neighbour of  $x$ .

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*AMS Subject Classification:* 60K37, 05C80, 60F20.

*Keywords:* random walk, random conductances, law of large numbers, spanning tree.

For any event  $A \subseteq (\mathbb{Z}^2)^\mathbb{N}$ , we define the annealed law to be

$$\mathbb{P}_z(A) = \int_{\Omega} P_z^\omega(A) P(d\omega).$$

If  $z$  is the origin, we may omit the superscript. We use  $\mathbb{E}_z$  and  $E_z^\omega$  as the expectations w.r.t. the distributions  $\mathbb{P}_z$  and  $P_z^\omega$ .

This is a well-known process, called the random walk among random conductances (RWRC). The RWRC has been studied extensively in the past decades, see e.g. [Bis11] and the references therein. It is related to many other important models in Statistical physics, for example the gradient fields (see e.g. [BS11]) and reinforced random walk (see e.g. [MR09, ACK12, ST12]).

Traditionally, RWRC is studied in connection to problems such as recurrence and transience [Ber02, ACK12, ST12], central limit theorems (see [BP07, Mat08, ABDH10] and many more) and heat kernel estimates (see e.g. [Del97, BB04, BBHK08]). However, in this paper we ask a slightly different question, that of the speed of escape from zero.

**1.2 Main question** It is easy to prove and well known that if the conductances are bounded then the speed is zero, i.e.  $\mathbb{P}(\lim(X_n/n) = 0) = 1$ . It is also well known that if the conductances are i.i.d. the same is true, regardless of the distribution of the single conductances. We thus wish to understand which conditions force the speed of the RWRC to be zero. Based on the examples above, it seems that two types of criteria are involved. The first is moment conditions that control the size of the conductances, and the second is mixing conditions saying that if the environment mixes fast enough then the speed is zero.

In this paper we only consider the first type, and show that the sharp condition is that the logarithm of the conductances has high enough moments.

Our main result is as follows.

thm:main

**Theorem 1.1.** *Let  $e$  be an edge in  $\mathbb{E}^2$ .*

item:carne

(i) *If there exists  $\alpha > 1$  such that*

$$E[\log^\alpha \omega_e] < \infty,$$

(1.1)

eq:logmomalpha

*then*

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0\right) = 1.$$

item:trees

(ii) *For every  $\alpha < 1$  there exists a distribution  $P$  on environments such that  $E[\log^\alpha \omega_e] < \infty$ , but*

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0\right) = 0.$$

Our proofs will deal with conductances bounded away from zero, but would work in the same way including the possibility of zero conductances. Note also that the choice of dimension 2 has been made in order to have easier and more

intuitive proofs. We are confident that the same results can be proven with the very same techniques in higher dimensions, with different critical  $\alpha$ 's.

In Section 2 we show Part 1 of Theorem 1.1, which end up being a simple application of the Varopoulos-Carne Theorem. In Sections 3 and 4 we show Part 2 of Theorem 1.1. The construction builds upon the example constructed by Bramson, Zeitouni and Zerner in [BZZ06].

## 2. MOMENT CONDITIONS FOR SPEED ZERO

**sec:carne**

In this section we prove Part 1 of Theorem 1.1.

In order to prove it, we will use the well known Varopoulos-Carne bound. For proof see, e.g., [Car85].

**lem:carne**

**Lemma 2.1** (Varopoulos-Carne). *Let  $L$  be an irreducible Markov transition kernel with reversible measure  $\pi$ . For states  $x$  and  $y$ , denote  $d(x, y) = \min\{n : L^n(x, y) > 0\}$ . Then for every  $x, y$  and  $n$ ,*

$$L^n(x, y) \leq 2\sqrt{\frac{\pi(y)}{\pi(x)}} \cdot e^{-\frac{d(x, y)^2}{2n}}. \quad (2.1)$$

**eq:carne**

*Proof of Part 1 of Theorem 1.1.* The measure  $\pi$  on  $\mathbb{Z}^2$ , defined by  $\pi(x) = \sum_{y \sim x} \omega_{\{x, y\}}$  is a reversible measure for our random walk. As in (1.1), let

$$D = E[\log^\alpha \omega_e] < \infty.$$

For  $n \in \mathbb{N}$ , consider the points  $x \in \mathbb{Z}^2$  such that  $\|x\|_\infty = n$ , and call  $E_n$  the set of edges having at least one end in these points. Note that  $|E_n| = 24n$ .

Then by Markov's inequality, for every  $n \in \mathbb{N}$  and  $K > 0$  we get

$$\mathbb{P}(\exists e \in E_n \text{ s.t. } \omega_e > \frac{K}{4}) \leq 24n \frac{D}{\log^\alpha(\frac{K}{4})}.$$

In particular, if  $K = e^{n^\beta}$  with  $1/\alpha < \beta < 1$ , then

$$\mathbb{P}(\exists e \in E_n \text{ s.t. } \omega_e > \frac{K}{4}) \leq Cn^{1-\alpha\beta},$$

for some constant  $C > 0$ .

Observe that  $1 - \alpha\beta < 0$ . Therefore, by Borel-Cantelli lemma, for an integer  $\kappa > (\alpha\beta - 1)^{-1}$ , a.s. for all  $n$  large enough and every edge  $e \in E_{n^\kappa}$ , we have

$$\omega_e < \frac{1}{4}e^{n^{\kappa\beta}}.$$

Therefore, for every  $x$  s.t.  $\|x\|_\infty = n^\kappa$ , we have that  $\pi(x) < e^{n^{\kappa\beta}}$ .

Now fix  $M \in \mathbb{N}$  and assume that  $M$  is large. For every  $n$  large enough,

$$\begin{aligned}
P^\omega(\|X_{Mn^\kappa}\|_\infty > n^\kappa) &\leq P^\omega(\exists k \leq Mn^\kappa : \|X_{Mn^\kappa}\|_\infty = n^\kappa) \\
&\leq \sum_{k=1}^{Mn^\kappa} \sum_{x: \|x\|_\infty = n^\kappa} P^\omega(X_k = x) \\
&\leq \sum_{k=1}^{Mn^\kappa} \sum_{x: \|x\|_\infty = n^\kappa} 2\sqrt{\frac{\pi(x)}{\pi(0)}} e^{-\frac{n^{2\kappa}}{2k}} \\
&\leq C' \pi(0)^{-1/2} \exp\left\{\frac{n^{\kappa\beta}}{2} - \frac{n^\kappa}{2M}\right\},
\end{aligned}$$

for some constant  $C' > 0$ .

Therefore, again by Borel-Cantelli, almost surely for all  $n$  large enough,

$$\|X_{Mn^\kappa}\|_\infty < n^\kappa.$$

From here we immediately get that almost surely

$$\limsup_{n \rightarrow \infty} \frac{\|X_n\|_\infty}{n} < \frac{2}{M}$$

and in fact, since  $M$  is arbitrary,

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0\right) = 1.$$

□

### sec:trees

### 3. TREES

In this section and in the next one we prove Part 2 of Theorem 1.1. The section is divided into two different subsections. In Subsection 3.1 we create the structure for the random environment where, with probability one, the sequence  $(\frac{X_n}{n})$  does not converge, and in Subsection 3.2 we create another example where with probability one the sequence  $(\frac{X_n}{n})$  converges to a speed which is not zero. In both cases  $E[\log^\alpha \omega_e] < \infty$  for arbitrary  $\alpha < 1$ . The example in Subsection 3.1 is a direct application of the tree construction of Bramson, Zeitouni and Zerner [BZZ06]. For the construction in Subsection 3.2, we need to modify the tree of [BZZ06]. The construction is inspired by the construction in [BZZ06], but we need to change quite a few details in order for the speed to converge.

In both cases, we adapt trees into environments for the random walk in the exact same fashion. This is done in Section 4. Now, we give a short introduction with the necessary terms from [BZZ06], and then, in Subsection 3.1 and 3.2, we create the actual trees.

An *ancestral function* is a (in our case random) function  $a : x \in \mathbb{Z}^2 \rightarrow a(x) \in \mathbb{Z}^2$  with the following properties:

- $x$  and  $a(x)$  are nearest neighbours;
- $a(a(x)) \neq x$ ;

- the set of edges  $F_a := \{\{x, a(x)\} : x \in \mathbb{Z}^2\}$  is a forest (i.e. the graph  $(\mathbb{Z}^2, F_a)$  contains no cycles).

Every connected component of  $F_a$  is an infinite tree.  $a(x)$  can be seen as the parent of  $x$  and we denote by  $a^n(x)$  the  $n$ -th generation ancestor of  $x$ , for  $n \geq 0$  (with the convention  $a^0(x) = x$ ).

We also say that an ancestral function is directed if for some  $i, j \in \{+1, -1\}$  and for every  $x \in \mathbb{Z}^2$ ,  $a(x) - x \in \{(0, i), (j, 0)\}$ .

The length of the longest branch starting in  $x$  (or the distance from  $x$  of its farthest descendant, if one prefers the genealogical metaphor) is

$$h(x) := \sup\{n \geq 0 : \exists y \in \mathbb{Z}^2 \text{ such that } a^n(y) = x\}. \quad (3.1)$$

eqn:bulk

We are interested in the distribution of  $h(0)$  in the case of a random translation invariant ancestral function.

Theorem 1 in [BZZ06] says that for any stationary ancestral function there exists a constant  $c \geq 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \mathbb{P}(h(0) \geq n) \geq c. \quad (3.2)$$

eqn:lower\_bound\_h

In the same article the authors show that this is in fact the best lower bound achievable. We give the 2-dimensional version of Theorem 2 in that paper:

thm:BZZ

**Theorem 3.1** ([BZZ06], Theorem 2). *There exists a stationary directed ancestral function  $(a(x))_{x \in \mathbb{Z}^2}$  that is polynomially mixing of order 1 and for which*

$$\limsup_{n \rightarrow \infty} n \mathbb{P}(h(0) \geq n) < \infty. \quad (3.3)$$

eqn:upper\_bound\_h

We now describe the BZZ tree, as appearing in [BZZ06].

sec:nospeed

### 3.1 The BZZ tree

We provide now the construction of the ancestral function used in [BZZ06], restricted to the 2 dimensional case. We will make use of the same notations as [BZZ06] with an additional tilde.

Let  $\{e_1, e_2\}$  be the canonical basis of  $\mathbb{Z}^2$ , with  $e_1$  parallel to the  $x$ -axis. Fix two constants  $\tilde{\theta}$  and  $\tilde{n}_0 \in \mathbb{N}$  such that  $2\sqrt{2} \leq \tilde{\theta} \leq \tilde{n}_0^2$ . For every  $x \in \mathbb{Z}^2$  let  $\tilde{L}(x)$  be i.i.d. random variables satisfying

$$\tilde{P}(\tilde{L}(x) > t) = \frac{\tilde{\theta}}{t^2} \quad \text{for } t \geq \tilde{n}_0. \quad (3.4)$$

We define an *umbrella* of intensity  $t$  to be

$$\tilde{U}_t = \bigcup_{i=1,2} \tilde{U}_{i,t} \quad (3.5)$$

where

$$\tilde{U}_{i,t} = \{y = (y_1, y_2) \in \mathbb{Z}^2 : y_i = 0, y_j \in (0, t], j \neq i\} \quad (3.6)$$

are the sides of the umbrella. The strength of the umbrella is also defined to be equal to its intensity.

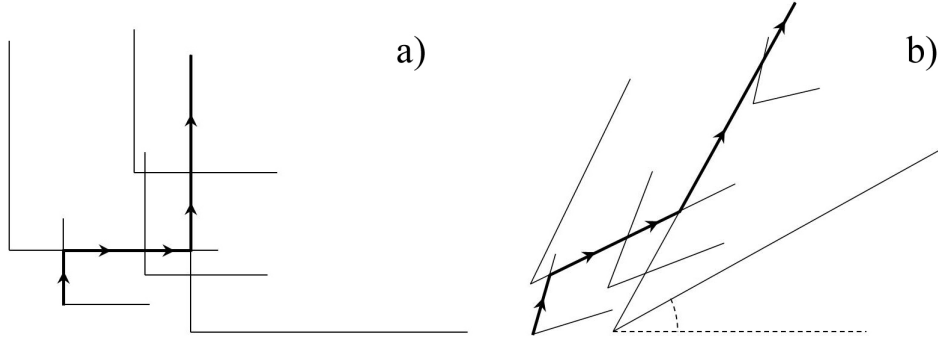


FIGURE 1. Both in the straight umbrellas case a) and in the narrow umbrellas case b) the drop of water follows the side of the biggest umbrella met. Note that in b) the longest umbrellas are also the ones that are the narrowest.

umbrellas

fig:umbrellas

For every  $x \in \mathbb{Z}^2$  we will open the umbrella  $x + \tilde{U}_{\tilde{L}(x)}$ . Informally, one can think of the ancestral function as a drop of rain trying to fall towards the up-right direction of the plane and sliding on the sides of the umbrellas. Whenever two or more umbrellas overlap, the water will consider only the strongest of them and penetrate the perpendicular ones.

Formally, one defines for every  $x \in \mathbb{Z}^2$  the strongest umbrella passing through that point in direction  $i \in \{1, 2\}$  as

$$\tilde{\lambda}_i(x) = \sup_{y \in \mathbb{Z}^2: x \in y + \tilde{U}_{i, \tilde{L}(y)}} \tilde{L}(y). \quad (3.7)$$

Note that the sup is taken over a non-empty set and it is easy to show that  $\tilde{\lambda}_i(x)$  is also a.s. finite.

Since the distributions of the  $\tilde{L}(x)$ 's are atomless, the direction  $I(x) \in \{1, 2\}$  such that

$$\tilde{\lambda}_{I(x)}(x) = \min\{\tilde{\lambda}_i(x), i = 1, 2\}$$

is well defined. The ancestral function we are looking for is

$$\tilde{a}(x) = x + e_{I(x)}. \quad (3.8)$$

The set of edges  $\{\{x, \tilde{a}(x)\}, x \in \mathbb{Z}^2\}$  through which the drops of rain have flown forms a random forest (which can be shown to be in fact a random tree spanning the whole  $\mathbb{Z}^2$ ). This is the ancestral function used to prove Theorem 3.1, and we will call the graph obtained with it the BZZ tree.

sec:speedpos

### 3.2 The Diagonal tree

We will now slightly modify the example seen in the previous subsection. Our aim is to build a new tree for which the behaviour of  $h(0)$  is essentially the same as in the BZZ tree, but with a different shape of the graph. Roughly speaking, it will not allow to have long strips that are "too horizontal" or "too vertical".

This feature and its importance will become more clear when we will describe the dynamics on these trees.

Fix the constants  $\theta$  and  $n_0 \in \mathbb{N}$  such that  $10 \leq \theta \leq n_0^2$ . For every  $x \in \mathbb{Z}^2$  consider i.i.d. random variables  $L(x) > 1$  with atomless distributions fulfilling

$$P(L(0) > t) = \frac{\theta \log t}{t^2} \quad \text{for all } t \geq n_0. \quad (3.9)$$

The new umbrellas we want to open are a bit different from the tilde-umbrellas of the previous section.

Define an umbrella of intensity  $t$  as

$$U_t = \bigcup_{i=1,2} U_{i,t} \quad (3.10)$$

where  $U_{1,t}$  is the best  $\mathbb{Z}^2$ -grid lower approximation of the open segment of length  $t$  that makes an angle of  $\frac{\pi}{4} - \frac{1}{\log t}$  with the  $x$ -axis, living in the first quadrant and starting in the origin.  $U_{2,t}$  is the reflection of  $U_{1,t}$  with respect to the bisecting line of the first quadrant.  $U_{1,t}$  and  $U_{2,t}$  are the sides of the umbrella. Note that this time the intensity gives us the strength, the length but also the width of the umbrella. In particular, the longer the umbrella, the more narrow it is.

We can think once more that drops of rain pouring from every point of the lattice try to fall towards the up-right direction and that every time they reach a new vertex, they are deflected by the strongest umbrella that passes through that vertex (see Figure 1).

In analogy with the straight-umbrellas case we define the strongest umbrella through  $x$  in direction  $i = 1, 2$  as

$$\lambda_i(x) = \sup_{y \in \mathbb{Z}^2: [x, x+e_i] \in y+U_L(y)} L(y). \quad (3.11)$$

eqn:lambda

Note that since  $L(0) > 1$  and since we are taking the lower (for the first component) and upper (for the second) approximations of the segments described above,  $[x, x+e_1] \in U_{1,L(x-e_1)}$  and  $[x, x+e_2] \in U_{2,L(x-e_2)}$ , so that the sup on the right hand side of (3.11) is taken over a non-empty set. It requires slightly more work compared to the straight-umbrellas case to prove that it is also a.s. finite and therefore well defined.

We need some more notations. Similarly to [BZZ06], for  $m, n \in \mathbb{Z}$  call  $S_m^n$  the slab

$$S_m^n = \{x = (x_1, x_2) \in \mathbb{Z}^2 : m \leq x_1 + x_2 \leq n\}.$$

The protecting area  $G$  (see Figure 2) is defined as

$$G := \left\{ x = (x_1, x_2) \in -\mathbb{N}^2 \mid \exists n \in \mathbb{N} : \right. \\ \left. x \in S_{-n}^{-n} \text{ and } x_1 \in \left[ y_n \cdot \cos\left(\frac{\pi}{4} - \alpha_n\right), y_n \cdot \cos\left(\frac{\pi}{4} + \alpha_n\right) \right] \right\}, \quad (3.12)$$

where  $\alpha_n = \arctan \frac{\sqrt{2}}{3 \log n}$  and  $y_n = \frac{n}{3\sqrt{2} \log n} \sqrt{2 + 9 \log^2 n}$ . These values guarantee that every segment  $S_{-m}^{-m} \cap G$  is  $\frac{2m}{3 \log m}$  long, and therefore contains  $\frac{\sqrt{2}m}{3 \log m}$  points of  $\mathbb{Z}^2$  (up to one unit, at most).

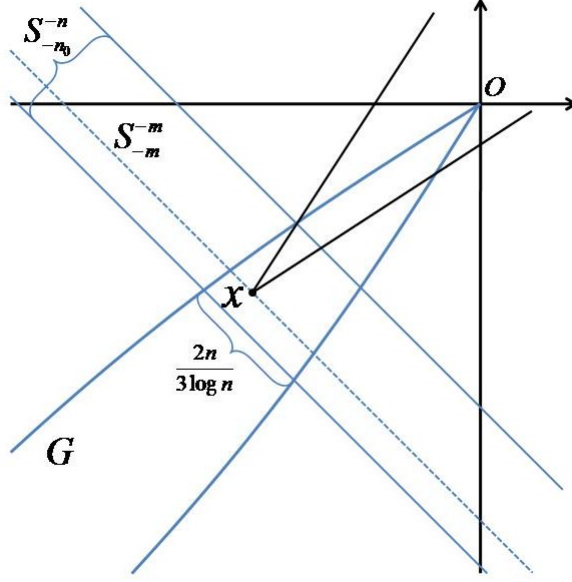


FIGURE 2. The protecting area  $G$  is the region of the plane from which we can have umbrellas that protect the origin. In particular, having a suitably strong umbrella starting in the part of  $G$  delimited by the slab  $S_{-n_0}^{-n}$  will ensure  $h(0) < n$  with high probability.

gz

Note that every umbrella  $x + U_s$  with  $x \in G$ ,  $-(x_1 + x_2) = n$  and  $s \in [n, n^2]$ , "protects" the origin  $O$ , meaning that  $O$  lies inside the " $\mathbb{Z}^2$ -triangle" generated by the sides  $x + U_{1,s}$  and  $x + U_{2,s}$ .

**Lemma 3.2.** *There is a constant  $c$  such that for  $i = 1, 2$  and  $t > n_0$ ,*

$$\mathbb{P}(\lambda_i(0) > t) \leq c \frac{\log t}{t}. \quad (3.13)$$

*Proof.* This is a straightforward calculation.

$$\begin{aligned} \mathbb{P}(\lambda_i(0) > t) &\leq C \int_t^\infty [s] \frac{\log s}{s^3} ds \\ &\leq C \sum_{k=0}^\infty \int_{2^k t}^{2^{k+1} t} s \frac{\log s}{s^3} ds \leq C \sum_{k=0}^\infty \frac{\log 2^k t}{2^k t} \\ &= C \frac{1}{t} \sum_{k=0}^\infty \frac{1}{2^k} [\log t + \log 2^k] \leq c \frac{\log t}{t}. \end{aligned}$$

□

Also in this case, the fact that the distributions of the  $L(x)$ 's are atomless guarantees the uniqueness of a direction  $I(x) \in \{1, 2\}$  such that

$$\lambda_{I(x)}(x) = \min\{\lambda_i(x), i = 1, 2\}.$$



For example, if  $I(x) = 1$ , it means that the strongest horizontal umbrella through  $x$  is weaker than the strongest vertical one.  $I(x)$  is the direction which the drop of water will follow.

We can therefore define the new ancestral function

$$a(x) = x + e_{I(x)}. \quad (3.14)$$

eqn:BSancestral

By its construction, it follows automatically that  $a : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  is stationary and directed.

thm:BS\_tree

**Theorem 3.3.** *The random ancestral function described in (3.14) is such that*

$$\limsup_{n \rightarrow \infty} \frac{n}{\log^2 n} \mathbb{P}(h(0) > n) < \infty. \quad (3.15)$$

### 3.3 Proof of Theorem 3.3

We closely follow the proof of Theorem 2 in [BZZ06].

We say that an umbrella  $U$  penetrates a weaker umbrella  $V$  in point  $x \in \mathbb{Z}^2$  if one side of  $U$  intersect one side of  $V$  and  $x$  is the upper-right point of their intersection. The following lemma bounds the probability that an umbrella of intensity  $t$  starting in the origin gets penetrated by another umbrella in a given point  $z$ .

mma:weak\_umbrella

**Lemma 3.4.** *Fix any  $t > n_0$ . Let  $z \in \mathbb{Z}^2$  such that  $[z, z + e_i] \in U_{j,t}$ , for some  $i, j \in \{1, 2\}$ . Then there exists a constant  $c > 0$  independent of  $t$  such that*

$$\mathbb{P}(I(z) = i | L(0) = t) \leq c \frac{\log t}{t}. \quad (3.16)$$

*Proof.* For convenience, we shift the umbrella so that  $z$  is translated to the origin. We look first at the event  $E_k$  that the umbrella gets penetrated in the origin by an umbrella of intensity  $s \in [k, k+1]$ , for  $k+1 > t$ . Note that such a penetrating umbrella can come only from  $S_{-k-1}^{-1}$ . Furthermore, on every  $S_m^m$ ,  $m \in \{-k-1, \dots, -1\}$ , there are almost surely at most four points that can generate it, since for all the others the slope of the sides would prevent them from penetrating the original umbrella in the origin. Hence

$$\mathbb{P}(E_k) \leq \sum_{j=1}^{k+1} 4 \left( \frac{\theta \log k}{k^2} - \frac{\theta \log(k+1)}{(k+1)^2} \right) \leq c' \frac{\log k}{k^2},$$

for some constant  $c'$ .

It is now easy to see that

$$\mathbb{P}(I(0) = i | L(-z) = t) \leq \sum_{k=\lfloor t \rfloor}^{\infty} \mathbb{P}(E_k) \leq c \frac{\log t}{t}.$$

□

For  $n \geq n_0$ , define now the random variables

$$M_n := \max \{m \in \{n_0, \dots, n\} : \exists x \in S_{-m}^{-m} \cap G \text{ with } m < L(x) < m^2\}, \quad (3.17)$$

with the convention  $M_n = n_0 - 1$  whenever the set on the right hand side is empty.

Proving that, for some constant  $c$ ,

$$\mathbb{P}(h(0) > m, M_n = m) \leq c \frac{\log^2 m}{m^2} \quad \forall m = n_0, \dots, n, \quad (3.18) \quad \boxed{\text{eqn:em\_en\_estimates}}$$

would imply

$$\mathbb{P}(h(0) > n) \leq \sum_{m=n_0-1}^n \mathbb{P}(h(0) > m, M_n = m) \leq c \frac{\log^2 n}{n}, \quad (3.19)$$

that is the statement of the theorem.

We first prove (3.18) in the easy case  $m = n_0 - 1$ .

$$\begin{aligned} \mathbb{P}(h(0) > m, M_n = n_0 - 1) &\leq \mathbb{P}(M_n = n_0 - 1) \\ &= \mathbb{P}(\text{for all } m = n_0, \dots, n \text{ and } x \in S_{-m}^{-m} \cap G, L(y) \notin (m, m+1]) \\ &= \prod_{m=n_0}^n (1 - \mathbb{P}(L(0) \geq m) + \mathbb{P}(L(0) > m^2))^{\#(S_{-m}^{-m} \cap G)} \\ &= \prod_{m=n_0}^n \left(1 - \frac{\theta \log m}{m^2} + \frac{\theta \log m^2}{m^4}\right)^{\frac{\sqrt{2}}{3} \frac{m}{\log m}} \\ &\leq \prod_{m=n_0}^n \left(1 - \theta \left(1 - \frac{2}{n_0^2}\right) \frac{\log m}{m^2}\right)^{\frac{\sqrt{2}}{3} \frac{m}{\log m}} \\ &\leq e^{-\theta \left(1 - \frac{2}{n_0^2}\right) \frac{\sqrt{2}}{3} \sum_{m=n_0}^n \frac{1}{m}} \leq c n^{-2} \end{aligned} \quad (3.20)$$

by the choice of  $\theta$ , for some  $c > 0$ .

For the more complex cases  $m = n_0, \dots, n$  we faithfully follow [BZZ06] once again. For  $m, n, r \in \mathbb{Z}$ ,  $x \in \mathbb{Z}^2$ , define the events

$$A_m^n(x, r) = \{L(y) \notin (-y \cdot \vec{1} + r, (-y \cdot \vec{1} + r)^2) \text{ for all } y \in S_m^n \cap (x + G)\}. \quad (3.21)$$

Firstly note that

$$\begin{aligned} \mathbb{P}(h(0) > m, M_n = m) &\leq \sum_{x \in S_{-m}^{-m} \cap G} \mathbb{P}(h(0) > m, L(x) \in (m, m^2), A_{m-n}^{-1}(0, 0)) \\ &= \sum_{x \in S_{-m}^{-m} \cap G} \mathbb{P}(h(-x) > m, L(0) \in (m, m^2), A_{m-n}^{-1}(-x, m)) \\ &= \sum_{x \in S_m^m \cap -G} \mathbb{P}(h(x) > m, L(0) \in (m, m^2), A_{m-n}^{-1}(x, m)), \end{aligned}$$

where we have used stationarity to obtain the second line and we write  $-G = \{x = (x_1, x_2) : (-x_1, -x_2) \in G\}$ .

Take now the segment  $S_m^m \cap -G$  and divide it in eight parts of the same lenght and call them  $I_1, \dots, I_8$ . For every  $j \in \{1, \dots, 8\}$ , consider  $\hat{x}^j$  and  $\check{x}^j$ , the points with respectively the highest and the lowest  $y$ -coordinate on  $I_j$ . Draw the cones  $\hat{C}^j$  and  $\check{C}^j$  with amplitude  $\frac{2}{\log m^2}$  whose bisector makes an angle of  $\frac{5}{4}\pi$  with the  $x$ -axis and with vertices  $\hat{x}^j$  and  $\check{x}^j$  respectively. Observe that the points in the area  $\hat{C}^j \cap \check{C}^j \cap S_{-n+m}^{-1}$  are contained in  $S_{-n+m}^{-m} \cap (x + G)$  for every  $x \in I_j$ . Therefore the event

$$E_j(m, n) := \left\{ L(y) \notin \left( -(y_1 + y_2) + m, -(y_1 + y_2) + m \right)^2 \right. \\ \left. \text{for all } y = (y_1, y_2) \in \hat{C}^j \cap \check{C}^j \cap S_{-n+m}^{-1} \right\}$$

is contained in the event  $A_{m-n}^{-1}(x, m)$  for all  $x \in I_j$ . Hence

$$\begin{aligned} & \sum_{x \in S_m^m \cap -G} \mathbb{P}(h(x) > m, L(0) \in (m, m^2), A_{m-n}^{-1}(x, m)) \\ & \leq \sum_{j=1}^8 \sum_{x \in I_j} \mathbb{P}(h(x) > m, L(0) \in (m, m^2), A_{m-n}^{-1}(x, m)) \\ & \leq \sum_{j=1}^8 \sum_{x \in I_j} \mathbb{P}(h(x) > m, L(0) \in (m, m^2), E_j(m, n)) \\ & = \sum_{j=1}^8 \mathbb{E} \left[ \#\{x \in I_j : h(x) > m\}; L(0) \in (m, m^2); E_j(m, n) \right]. \end{aligned} \quad (3.22) \quad \boxed{\text{eqn:first\_part}}$$

The interval  $(m, m^2)$  can be divided in a finite number of disjoint subintervals such that the  $\mathbb{Z}^2$  approximation of every umbrella with intensity in a given subinterval looks the same at least up to the first  $m$  edges. More precisely, there exists  $M \in \mathbb{N}$  and there exist  $\{m_1 = m < m_2 < \dots < m_M = m^2\}$  such that, for any  $k \in \{1, 2, \dots, M\}$ ,  $\forall h, l \in (m_k, m_{k+1})$ , one has  $U_h|_m = U_l|_m$ , where  $U_h|_m$  is the umbrella of intensity  $h$  whose sides are restricted to the first  $m$  edges (going from bottom-left towards up-right). Therefore, we can rewrite (3.22) as

$$\sum_{j=1}^8 \sum_{l=1}^M \mathbb{E} \left[ \#\{x \in I_j : h(x) > m\}; L(0) \in (m_l, m_{l+1}); E_j(m, n) \right]. \quad (3.23) \quad \boxed{\text{eqn:first\_part2}}$$

For any point  $x \in S_m^m \cap -G$  to have  $h(x) > m$ , there must be a branch coming out of  $x$  that perforates the protecting umbrella generated by the origin (since  $L(0) \in (m, m^2)$ ). That is, at least one point  $z$  on  $U_{L(0)}|_m$  must be penetrated by another umbrella. On the other hand, every penetrated  $z$  can give rise to at most one of such  $x$ 's. Hence, for any  $l = 1, \dots, M$ , given  $L(0) \in (m_l, m_{l+1})$ ,

$$\#\{x \in S_m^m \cap -G : h(x) > m\} \leq \sum_{i=1,2} \sum_{[z, z+e_i] \in U_{L(0)}|_m} \mathbb{1}_{\{I(z)=i\}}. \quad (3.24)$$

Plugging this in (3.23) gives

$$\begin{aligned} & \mathbb{P}(h(0) > m, M_n = m) \\ & \leq \sum_{j=1}^8 \sum_{l=1}^M \sum_{i=1,2} \sum_{[z, z+e_i] \in U_{L(0)}|_m} \mathbb{P}(I(z) = i, L(0) \in (m_l, m_{l+1}), E_j(m, n)). \end{aligned}$$

The intersection of the first two events inside the last probability is not independent of  $E_j(m, n)$ , but there is a negative correlation between them. We obtain therefore the upper bound

$$\begin{aligned} & \mathbb{P}(h(0) > m, M_n = m) \\ & \leq \sum_{j=1}^8 \sum_{l=1}^M \sum_{i=1,2} \sum_{[z, z+e_i] \in U_{L(0)}|_m} \mathbb{P}(I(z) = i, L(0) \in (m_l, m_{l+1})) \mathbb{P}(E_j(m, n)). \end{aligned}$$

We can now directly compute the right hand side of last expression. For  $[z, z+e_i] \in U_{L(0)}|_m$  we have, by Lemma 3.4,

$$\begin{aligned} \mathbb{P}(I(z) = i; L(0) \in (m_l, m_{l+1})) &= \int_{m_l}^{m_{l+1}} \mathbb{P}(I(z) = i | L(0) = t) \left( \frac{d}{dt} \mathbb{P}(L(0) \leq t) \right) dt \\ &\leq \int_{m_l}^{m_{l+1}} c \frac{\log t}{t} \frac{\theta}{t^3} (2 \log t - 1) dt \\ &\leq K \frac{\log^2 m}{m^4} (m_{l+1} - m_l), \end{aligned} \tag{3.25}$$

for some constant  $K$ .

Summing over the directions  $i = 1, 2$  and over all the  $z \in \mathbb{Z}^2$  such that  $[z, z+e_i] \in U_{L(0)}|_m$  and then summing over  $l = 1, \dots, M$ , one is left with a factor of order  $\frac{\log^2 m}{m^2}$ .

In order to evaluate the probability of any  $E_j(m, n)$ , note that every  $E_j \cap S_{-k}^{-k}$  contains more than  $\frac{1}{4} \frac{k}{\log k}$  points of the lattice. In fact, each cone  $\hat{C}^j$  and  $\check{C}^j$  intersects  $S(k)$ , the hyperplane containing  $S_{-k}^{-k}$ , on the segments  $\hat{H}_k^j$  and  $\check{H}_k^j$ , each of lenght bigger than  $\frac{2}{3} \frac{k}{\log k}$  (it is the double of the lenght of the cathetus of a right triangle, whose opposite angle measures  $\frac{1}{\log(m^2)}$  degrees and with the other cathetus  $\frac{m}{\sqrt{2}}$  long). The intersection of  $\hat{H}_k^j$  and  $\check{H}_k^j$  is therefore longer than  $(\frac{1}{3} + \frac{1}{3} - \frac{3}{16}) \frac{k}{\log k} = \frac{23}{48} \frac{k}{\log k}$ . Since the distance between close points on  $E_j \cap S_{-k}^{-k}$  is  $\sqrt{2}$ , the total number of points is bigger than  $\frac{1}{\sqrt{2}} \frac{23}{48} \frac{k}{\log k} \geq \frac{1}{4} \frac{k}{\log k}$ .

By the independence of the  $(L(x))_{x \in \mathbb{Z}^2}$

$$\begin{aligned}
\mathbb{P}(E_j(m, n)) &\leq \prod_{k=m+1}^n \left(1 - \frac{\theta \log k}{k^2}\right)^{\frac{1}{4} \frac{k}{\log k}} \\
&\leq \exp \left\{ -\frac{\theta}{4} \sum_{k=m+1}^n \frac{1}{k} \right\} \\
&\leq \exp \left\{ -\frac{\theta}{4} \int_{m+1}^n \frac{1}{s} ds \right\} \\
&= \left( \frac{n}{m+1} \right)^{-\frac{\theta}{4}}.
\end{aligned} \tag{3.26}$$

Putting all together we finally obtain, for some constant  $c$ ,

$$\begin{aligned}
\mathbb{P}(h(0) > m, M_n = m) &\leq c \frac{\log^2 m}{m^2} \left( \frac{n}{m+1} \right)^{-\frac{\theta}{4}} \\
&\leq c(m+1)^{\frac{\theta}{4}-2} n^{-\frac{\theta}{4}} \log^2 m \\
&\leq c n^{-2} \log^2 n.
\end{aligned} \tag{3.27}$$

sec:env

#### 4. THE ENVIRONMENT

The two random trees constructed in the previous sections will provide, in some sense, the support for our random environments. In both cases, the  $\omega$ 's are constructed in the following way.

Sample a realization of the tree as described above. For every  $z \in \mathbb{Z}^2$ , the edge  $\{z, a(z)\}$  will have a conductance value of  $\omega_{\{z, a(z)\}} = e^{(h(z)+1)^A}$ , where  $a : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  is the ancestral function used for constructing the sampled tree and  $A > 1$  is a constant. We set all the other conductances to be equal to one.

For both the BZZ and the Diagonal tree, the conductances have infinite  $\alpha$ -logmoments for any  $\alpha > 1$ . On the other hand, choosing appropriately the constant  $A > 1$ , we can obtain conductances with finite  $\alpha$ -logmoments for  $\alpha$  arbitrarily close to 1 from below.

**Proposition 4.1.** *Take  $\bar{\alpha} < 1$ . Then, the conductances of the random environments described above with  $1 < A < \frac{1}{\bar{\alpha}}$  are such that*

$$E[\log^\alpha \omega_e] \leq \infty \quad \forall \alpha \leq \bar{\alpha} \tag{4.1}$$

and

$$E[\log^\alpha \omega_e] = \infty \quad \forall \alpha \geq 1. \tag{4.2}$$

*Proof.* We first prove it for the random environment built on the BZZ-tree support.

$$\begin{aligned}
E[\log^\alpha \omega_e] &= \int_0^\infty P(\log^\alpha \omega_e > t) dt \\
&= \sum_{k=0}^\infty \int_{k^{\alpha A}}^{(k+1)^{\alpha A}} P(\log^\alpha \omega_e > t) dt \\
&\leq \sum_{k=0}^\infty P(\log^\alpha \omega_e > k^{\alpha A}) ((k+1)^{\alpha A} - k^{\alpha A}) \\
&= \sum_{k=0}^\infty P(h(0) > k) ((k+1)^{\alpha A} - k^{\alpha A}) \tag{4.3}
\end{aligned}$$

eqn:logmoments\_series

By equations (3.2) and (3.3) we know that for all sufficiently large  $k \in \mathbb{N}$ , say  $k \geq K$ ,

$$\frac{c}{k} \leq P(h(0) > k) \leq \frac{c'}{k}.$$

Therefore, on the one hand, taking  $\alpha \leq \bar{\alpha}$ ,

$$E[\log^\alpha \omega_e] \leq C + \sum_{k=K}^\infty \frac{c'}{k} \alpha A (k+1)^{\alpha A-1} < \infty,$$

where  $C > 0$  is the finite contribution of the first  $K-1$  terms of the sum, while, on the other hand, when  $\alpha \geq 1$  we obtain, with a minor modification of (4.3),

$$E[\log^\alpha \omega_e] > \sum_{k=K}^\infty \frac{c}{k} \alpha A k^{\alpha A-1} = \infty.$$

Note that the very same proof is valid for the random environment built over the Diagonal tree structure, since the  $\log^2$ -correction in Theorem 3.3 doesn't change the behaviour of the series (4.3).  $\square$

prop:follow\_tree

**Proposition 4.2.** *For almost every environment  $\omega$  sampled from the constructions of the previous section, the random walk among the conductances  $\omega$  will eventually follow the tree. This means that almost surely there exists  $\bar{n} < \infty$  such that for all  $n \geq \bar{n}$ , if  $X_n = x$  then  $X_{n+1} = a(x)$ , where  $a : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  is the ancestral function used to construct the tree underlying the environment.*

*Proof.* The probability that, starting in a point  $x \in \mathbb{Z}^2$ , the random walk will follow the tree forever is, by the independence of the jumps, bigger than

$$\prod_{k=1}^\infty \frac{e^{k^A}}{2e^{(k-1)^A} + e^{k^A} + 1}. \tag{4.4}$$

eqn:follow\_tree

It is easy, in fact, to get convinced that this is a very pessimistic estimate. It represents the case in which we start from a leaf of the tree (that is, a vertex that is ancestor of no other vertices) and where every time  $\omega_{\{X_n, a(X_n)\}}$  is of order  $k$  (that is, equal to  $e^{k^A}$ ), then the two edges under and at the left of  $X_n$  are of order  $k-1$ .

Call  $T_1, T_2, \dots$  the times in which the random walk doesn't go in the direction of the ancestral function. After each of these times, a new attempt to follow the tree is performed. Therefore if we show that the product (4.4) is a constant strictly bigger than zero, then the sum of the probabilities of succeeding in following the tree in one of the attempts is infinite. By Borel-Cantelli lemma, this means that almost surely there will be a finite time from which we will always follow the tree.

We are left to show that (4.4) is bigger than zero, or, equivalently, that its log is bigger than  $-\infty$ :

$$\begin{aligned} \log \left( \prod_{k=1}^{\infty} \frac{e^{k^A}}{2e^{(k-1)^A} + e^{k^A}} \right) &= - \sum_{k=1}^{\infty} \log \left( 1 + 2 \frac{e^{(k-1)^A}}{e^{k^A}} \right) \\ &> -2 \sum_{k=1}^{\infty} e^{(k-1)^A - k^A} \\ &> -2A \sum_{k=1}^{\infty} e^{-(k-1)^{A-1}} > -\infty, \end{aligned} \quad (4.5)$$

where we have used the mean value theorem for the bound  $k^A - (k-1)^A \geq A(k-1)^{A-1}$ .

□

prop:bzzspeed

**Proposition 4.3.** *The random walk among random conductances with environment built on the BZZ tree, as described above, has almost surely no limiting speed.*

prop:diagspeed

**Proposition 4.4.** *The random walk among random conductances with environment built on the diagonal tree, as described above, has almost surely a limiting speed which is not zero.*

*Proof of Proposition 4.3.* From Proposition 4.2 we know that with probability 1 there exists a finite time from which the random walk will use only edges pointing the right or up direction with respect to its current position. Without loss of generality we can think this time to be time 0. In order to study the limiting speed of the process, we have to go back to the underlying structure of the tree on which we have built the environment. Note that every time the random walk makes a step in the direction of the ancestral function, it finds several new umbrellas perpendicular to its previous step and a new parallel one. If the strongest perpendicular umbrella is stronger than any other umbrella on the direction of the previous step, the branch of tree changes orientation and the next step of the random walk will follow it; otherwise, it will perform another step in the same direction as before.

The distribution of the lenght  $\bar{L}$  of the new perpendicular umbrellas met at each step is easy to calculate:

$$\begin{aligned}
\mathbb{P}(\bar{L}(0) > t) &= \mathbb{P}(\exists j \in \mathbb{N} \text{ such that } \tilde{L}((0, -j)) > \max\{t, j\}) \\
&= 1 - \prod_{j=1}^{\lfloor t \rfloor} \mathbb{P}(\tilde{L}((0, -j)) \leq t) \prod_{j=\lfloor t \rfloor + 1}^{\infty} \mathbb{P}(\tilde{L}((0, -j)) \leq j) \\
&= 1 - \left(1 - \frac{\tilde{\theta}}{t^2}\right)^{\lfloor t \rfloor} \prod_{j=\lfloor t \rfloor + 1}^{\infty} \left(1 - \frac{\tilde{\theta}}{j^2}\right)
\end{aligned} \tag{4.6}$$

so that, by straightforward calculations,

$$\frac{c'}{t} \leq \mathbb{P}(\bar{L}(0) > t) \leq \frac{c''}{t}, \tag{4.7}$$

for some  $c', c'' > 0$ .

Now, being on a branch of the tree, what is the probability of passing from the umbrella that has generated that part of the branch to a stronger one before the umbrella itself ends? Suppose that the random walk is on a branch of the tree generated by an umbrella of lenght  $k > 2n_0^2$ . Then the probability of not meeting a stronger perpendicular umbrella or a stronger umbrella on the same direction of the current one before leaving the present umbrella is bigger than

$$\left(1 - \frac{c''}{k}\right)^k \left(1 - \frac{\tilde{\theta}}{k^2}\right)^k > e^{-2c''-1},$$

that is a constant strictly smaller than 1 and independent of  $k$ .

Considering only the strongest umbrellas through each point, call *rush* a sequence of intersecting umbrellas each bigger of the previous one that determines a part of the final tree.

Starting on any rush, the probability of leaving it (that is, of travelling the whole length of one of the umbrellas without meeting a stronger one) after having visited  $N \in \mathbb{N}$  different umbrellas is

$$\mathbb{P}(\text{Leave the rush after more than } N \text{ umbrellas}) < (1 - ce^{-2c''})^N, \tag{4.8}$$

for some  $c > 0$ .

This means, by Borel-Cantelli lemma, that with probability 1 the random walk will leave any rush in finite time. Given a realization of the walk, call  $\tau(1) \in \mathbb{N}$  the time in which the random walk leaves the first rush,  $\tau(2)$  the time in which it leaves the second one and so on.  $\tau(1) < \tau(2) < \dots$  is a sequence of (almost surely finite) integer stopping times that goes to infinity.

Fix  $T > 1$  and define the times  $T_1 = T$ ,  $\tau_1 = \min_{i=1,2,\dots} \{\tau(i) : \tau(i) > T_1\}$  and recursively

$$\begin{aligned}
T_k &= \tau_{k-1} + \tau_{k-1} T^k & \forall k > 1, \\
\tau_k &= \min_{i=1,2,\dots} \{\tau(i) : \tau(i) > T_{k-1}\} & \forall k > 1.
\end{aligned} \tag{4.9}$$



Our aim is now to show that in the intervals of the form  $(T_{k-1}, T_k)$ , the longest umbrella met is of length of the order  $\tau_{k-1}T^k$ . We don't want the longest umbrella to be much longer than this, otherwise it could "interfere" with the next intervals: consider the event

$$E_k = \{\text{In the interval } (T_{k-1}, T_k) \text{ the longest umbrella met is stronger than } T^{k+1}\tau_k\}.$$

Its probability can be bounded from above by

$$\begin{aligned} \mathbb{P}(E_k) &< 1 - \left(1 - \frac{\tilde{\theta}}{\tau_k T^{k+1}}\right)^{T_k} \\ &< 1 - e^{-2\frac{T_k}{\tau_k} T_k} \\ &< \frac{c}{T^{k+1}}, \end{aligned}$$

for some constant  $c > 0$ , since  $T_k \leq \tau_k$ . By Borel-Cantelli lemma,  $\mathbb{P}(E_k \text{ i.o.}) = 0$ .

On the other hand, we don't want the longest umbrella to be shorter than that. This is because we want it to be long a positive fraction of the entire time interval  $(T_{k-1}, T_k)$ . In fact, the interval  $(T_{k-1}, T_k)$  is long about  $\tau_{k-1}T^k$ . Furthermore, we want the random walk to follow this umbrella for a positive fraction (say an  $\varepsilon > 0$  fraction) of its length before leaving the time interval. This two events guarantee a relevant contribution to the speed up to time  $T_k$ . Therefore take, for a fixed  $\varepsilon > 0$  small,

$$F_k = \{\text{In the interval } (T_{k-1}, T_k(1 - \varepsilon)) \text{ the longest umbrella met is stronger than } \tau_{k-1}T^k \text{ and is bigger than the biggest umbrella in } (T_k(1 - \varepsilon), T_k)\}.$$

By the independence of the new umbrellas discovered at each step, we have, for all  $k \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{P}(F_k) &> (1 - \varepsilon)\mathbb{P}(\text{one of the } T_k - T_{k-1} \text{ umbrellas is longer than } \tau_{k-1}T^k) \\ &= (1 - \varepsilon)\left(1 - \left(1 - \frac{c'}{\tau_{k-1}T^k}\right)^{T_k - T_{k-1}}\left(1 - \frac{\tilde{\theta}}{\tau_{k-1}^2 T^{2k}}\right)^{T_k - T_{k-1}}\right) \\ &> (1 - \varepsilon)\left(1 - \left(1 - \frac{c'}{\tau_{k-1}T^k}\right)^{T_k - T_{k-1}}\right) \\ &> (1 - \varepsilon)\left(1 - e^{-\frac{c'}{2} \frac{1}{\tau_{k-1}T^k} (\tau_{k-1} + \tau_{k-1}T^k)}\right) \\ &= C \end{aligned} \tag{4.10}$$

where  $C > 0$  is a constant not depending on  $k$ . By the second Borel-Cantelli lemma, there are almost surely infinitely many intervals  $(T_{k-1}, T_k)$  for which  $F_k$  happens.

Hence, almost surely there exists a  $\bar{k} \in \mathbb{N}$  (depending eventually on the realization of the environment and of the random walk) such that  $E_k$  does not happen for every  $k > \bar{k}$  while  $F_k$  holds infinitely many times. Take now the strongest umbrella met up to time  $T_{\bar{k}}$ . Its length  $L > 0$  is almost surely finite, so that

$\kappa := \min\{k : T_k > T_{\bar{k}} + L\}$  is well defined. Note that  $\forall k > \kappa + 1$ , in the interval  $(T_k, T_{k+1})$  there is no umbrella longer than  $T^{k+1}\tau_k$  met in the past.

Take the infinite subsequence  $\kappa < k_1 < k_2 < \dots$  such that  $F_{k_i}$  holds true for every  $i \in \mathbb{N}$  and such that the longest umbrella met in the  $k_i$ 'th interval  $(T_{k_i-1}, T_{k_i})$  is followed by the random walk at least for a positive fraction  $0 < \eta < \epsilon$  of its length. Note that since there is no longer umbrella coming from a previous interval, once the random walk meets this umbrella it follows it until its end or at least until the end of the interval itself, and the probability of meeting the umbrella before the last  $\eta$  fraction of its length is strictly positive. This implies that we have such a sequence  $(k_i)_{i=1,2,\dots}$  almost surely.

Suppose now that a limiting speed  $v = (v[1], v[2])$  existed. We want to show that in each of those intervals there is at least one time  $t$  at which the ratio  $\sum_{j=1}^t X_j/t$  is far from  $v$ , bringing to a contradiction. Call  $t_i \in \mathbb{N}$  the time at which the longest umbrella of the interval  $(T_{k_i-1}, T_{k_i})$  is met and  $t^i = \{\text{"Time of the last point of the umbrella"} \wedge T_{k_i}\}$ . By definition, this umbrella is longer than  $\tau_{k_i-1}T^{k_i}$ , it is met before time  $T_{k_i}(1 - \epsilon)$  and before the last  $\eta$ -fraction of its length. Call

$$\frac{1}{t_i} \sum_{j=1}^{t_i} X_j = (v_i[1], v_i[2]) =: v_i$$

and

$$\frac{1}{t^i} \sum_{j=1}^{t^i} X_j = (v^i[1], v^i[2]) =: v^i$$

the partial speeds up to time  $t_i$  and  $t^i$  respectively. Without loss of generality suppose that we met the longest umbrella on its horizontal side. Note that

$$v_i[1] = \frac{1}{t_i} (v^i[1]t^i - t^i + t_i)$$

and that

$$\frac{t^i}{t_i} > 1 + \frac{\eta\tau_{k_i-1}T^{k_i}}{(1 - \epsilon)\tau_{k_i-1}(T^{k_i} + 1)} > 1 + \frac{\eta}{2(1 + \epsilon)} =: \beta > 0.$$

Further suppose  $v[1], v[2] \notin \{0, 1\}$ . Then if  $v[1] > v^i[1]$

$$|v[1] - v_i[1]| = v[1] - v^i[1]\frac{t^i}{t_i} + \frac{t^i}{t_i} - 1 > (\beta - 1)(1 - v[1]) > 0, \quad (4.11)$$

while if  $v[1] \leq v^i[1]$

$$\begin{aligned} \max\{|v[1] - v^i[1]|, |v[1] - v_i[1]|\} &\geq \frac{1}{2}(v^i[1] - v_i[1]) \\ &= \frac{1}{2}(v^i[1] - v^i[1]\frac{t^i}{t_i} + \frac{t^i}{t_i} - 1) \\ &> \frac{1}{2}(\beta - 1)(1 - v[1]). \end{aligned} \quad (4.12)$$

In both cases the distance from the limiting speed is bigger than a constant that is independent of  $k_i$  and strictly bigger than zero.

The cases  $v[1] = 1$  and  $v[1] = 0$  have probability 0. In fact, the probability of meeting in any interval  $(T_{k_i-1}, (1 - \epsilon)T_{k_i})$  a vertical (respectively, horizontal)

umbrella of order  $\tau_{k_i-1}T^{k_i}$  that is stronger of any other horizontal (vertical) umbrellas met before (and of following it for a time of  $O(t)$ ) is strictly positive, for the reasons mentioned above.

□

*Proof of Proposition 4.4.* Let  $v = (0.5, 0.5)$ . We claim that, almost surely,

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = v.$$

As in the previous proof, let  $N_0$  be such that for every  $n > N_0$ , we have  $X_{n+1} = a(X_n)$ , where  $a$  is the ancestral function. By a minor modification of Proposition 4.2,  $N_0$  is almost surely finite. We need to prove that for every  $\varepsilon > 0$  there exists a (random) finite  $M$  such that for every  $n > M$ , we have  $\|X_n/n - v\| < \varepsilon$ , where we write  $\|\cdot\|$  for, e.g., the usual  $\ell^1$ -norm. To this end, we need to understand the various umbrellas that the random walk traverses. By the construction of the diagonal tree, there exists  $K > 0$  such that for every umbrella which is stronger than  $K$ , for every two points  $x$  and  $y$  on the umbrella whose distance is larger than some  $U = U(\varepsilon)$ , we have

$$\left\| \frac{y - x}{\|y - x\|} - v \right\| < \varepsilon.$$

For umbrellas which are not stronger than  $K$ , their distribution is symmetric w.r.t. the diagonal, and their directions are i.i.d. and therefore they give an average of  $v$ .

Therefore,

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = v.$$

□

## 5. ACKNOWLEDGMENT

The second author would like to thank Simon Wasserroth from TU Berlin for useful discussions on worms. The research of the first author was partially supported by the Humboldt foundation and by ERC grant StG 239990.

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